

① Show $X = \left\{ \left(x_k \right)_{k=1}^{\infty} \in \mathbb{R}^N \mid \exists N \text{ s.t. } \forall k, k > N \Rightarrow x_k = 0 \right\}$

$$d(x_k, y_k) = \sup_k |x_k - y_k|$$

is not a complete metric space.

Sohn: Understand X :

Sequences in X

1, 2, 3, 4, 0, 0, 0 ...

0, 0, 0, 0, ...

1, 1/2, 1/3, 1/4, 0, 0, 0, 0

1, 1/2, 1/3, 1/4, 1/5, 0, 0, 0 ...

Sequences not in X

1, 1/2, 1/3, 1/4, 1/5, 1/6, ...

1/2, 1/4, 1/8, 1/16 ...

1, 2, 3, 4, ...

To show X is not complete, produce a Cauchy sequence in X with no limit.

$$\text{let } \{x_n^{(k)}\} = \begin{cases} 1/n & n \leq k \\ 0 & n > k \end{cases}$$

$$x_n^{(1)} = 1, 0, 0, 0, \dots$$

$$x_n^{(2)} = 1, \frac{1}{2}, 0, 0, \dots$$

$$x_n^{(3)} = 1, \frac{1}{2}, \frac{1}{3}, 0, \dots$$

:

Claim: $\{x_n^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence (of sequences) in X

Pf: Assume $k > j$, then

$$x_n^{(k)} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{j}, \frac{1}{j+1}, \dots, \frac{1}{k-1}, \frac{1}{k}, 0, 0, \dots$$

$$x_n^{(j)} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{j}, 0, \dots, 0, 0, 0, 0, \dots$$

}

$$\begin{aligned}
 d(x_n^{(k)}, x_n^{(j)}) &= \sup_n \{|x_n^{(k)} - x_n^{(j)}|\} \\
 &= \sup \left\{ \underbrace{0}_{\uparrow}, \frac{1}{j+1}, \frac{1}{j+2}, \dots, \frac{1}{k-1}, \frac{1}{k} \right\} \\
 &\quad x_n^{(k)} = x_n^{(j)} \\
 &\quad \text{when } n \leq j \\
 &= \frac{1}{j+1} \rightarrow 0 \quad \text{as } j \rightarrow \infty
 \end{aligned}$$

So this sequence is Cauchy.

This sequence does not converge in X because

i.e. if $x_n^{(k)}$ converges to something as $k \rightarrow \infty$ it must be the sequence
 $1, 1/2, 1/3, \dots$

)

which is not in X . (Exercise to prove this!)

② Let $\tilde{X} = \left\{ (x_n)_{n=1}^{\infty} \mid x_n \rightarrow 0 \text{ as } n \rightarrow \infty \right\}$ with same metric d from ①

Prop: (\tilde{X}, d) is a complete metric space.

Proof: . Exercise to show d is a metric.

. Completeness. Need to show any Cauchy sequence converges.

Lemma: If $\left\{ \left\{ x_n^{(k)} \right\}_{k=1}^{\infty} \right\}_{n=1}^{\infty}$ is Cauchy then $\forall n \exists x_n \in \mathbb{R} \quad x_n^{(k)} \xrightarrow[k \rightarrow \infty]{} x_n$

Proof: Fixing any n , the condition $\sup_n |x_n^{(k)} - x_n^{(j)}| \xrightarrow{j, k \rightarrow \infty} 0$

implies $\{x_n^{(k)}\}_{n=1}^{\infty}$ is Cauchy as a sequence in \mathbb{R} .

This converges, by completeness of \mathbb{R} , to some element $x_n \in \mathbb{R}$. \square

Returning to proof of prop...

For each n , produce the x_n s.t. $x_n^{(k)} \xrightarrow{k \rightarrow \infty} x_n$ and consider the sequence $\{x_n\}_{n=1}^{\infty}$. We claim $x_n \xrightarrow{n \rightarrow \infty} 0$. Fix $\varepsilon > 0$

$$|x_n| \leq |x_n - x_n^{(k)}| + |x_n^{(k)}| \quad \text{Triangle ineq.}$$

$$\leq \varepsilon \quad \begin{aligned} &\text{Choose } k \text{ large so } |x_n - x_n^{(k)}| \leq \varepsilon/2 \\ &\text{and } |x_n^{(k)}| \leq \varepsilon/2. \end{aligned}$$

Clearly $\{x_n^{(k)}\} \xrightarrow{k \rightarrow \infty} \{x_n\}$ in \widehat{X} so we have produced a limit to this Cauchy sequence.

□

- ③ Show $[1, \infty) \subseteq \mathbb{R}$ with $d(x,y) = |\log(\frac{x}{y})|$ is complete, but not

compact.

Pf: We will show every Cauchy sequence in \mathbb{d} is Cauchy in the Euclidian norm.

Choose a Cauchy sequence $\{x_k\}_{k=1}^{\infty} \subseteq [1, \infty)$.

Since Cauchy sequences are bounded, $\exists M$ s.t. $\{x_k\} \subseteq [1, M]$

By mean value theorem, $\forall j, k \quad \exists c \in [1, M]$ s.t.

$$|\log x_k - \log x_j| = \left| \frac{1}{c} \right| |x_k - x_j|$$

Notice $1 \leq c \leq M$, so

$$|x_k - x_j| \geq |\log x_k - \log x_j| \geq \frac{1}{M} |x_k - x_j|$$

This inequality

$|x_k - x_j| \geq \frac{1}{M} |x_k - x_j|$

this inequality implies $\{x_n\}$ is
Cauchy. $\left[\text{Indi} \right] \quad s$

guarantees $x_n \rightarrow x$
in the metric d .

Cauchy in Euclidian norm, so
converges in Euclidian norm to x .

To show $[1, \infty)$ is not compact, produce a sequence with no convergent subsequences.

Check that $x_k = k$ works, because $\log k \rightarrow \infty$

④ let $S \subseteq X$

$$\overline{S} = \left\{ x \in X \mid \exists \{x_k\} \subseteq S, x_k \rightarrow x \right\} := B$$

Definition: $x \in \overline{S} \Leftrightarrow \forall A \ni S$, if A closed then $x \in A$.

($\overline{S} \ni B$) Choose any $x \in B$. Clearly $S \subseteq \overline{S}$, and by definition x is a limit point of S (and thus also of \overline{S}). Since \overline{S} is closed as an intersection of closed sets, $x \in \overline{S}$.

as an intersection of closed sets, $x \in S$.

(\subseteq) Fix $x \in \overline{S}$. If $x \in S$ there is nothing to show.

If $x \notin S$ then $x \in \partial S$; indeed, if $\exists \varepsilon > 0$ $B(\varepsilon, x) \subseteq S^c$

then $B(\varepsilon, x)^c$ is a closed set containing S which does not contain x , contradicting the defn of $x \in \overline{S}$.

So $\forall \varepsilon > 0$, $B(\varepsilon, x) \cap S \neq \emptyset$

Choose $\varepsilon = 1/n$ and pick $x_n \in B(1/n, x) \cap S$.

Clearly $\{x_n\} \subseteq S$ is a sequence converging to x .